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A complete set of Bianchi identities for tensor fields along the tangent bundle projection

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Abstract. The various derivations defined along the tangent bundle projection τ in a series of papers by Martínez, Cariñena and Sarlet are expressed as components of a single linear connection $\bar{\nabla}$ on E, the tangent bundle of the evolution space $E = \mathbf{R} \times TM$. This connection is equivalent to a system of second-order ordinary differential equations (SODE) on M. Using the linear connection, we calculate the torsion and curvature of $(E, \bar{\nabla})$, the components of which are expressed in terms of the tensors along τ defined by Martínez *et al.* From these, the full set of Bianchi identities are calculated. We also show that the generalized Jacobi equation, defined by several authors, is precisely the horizontal component of the conventional Jacobi equation along geodesics of $(E, \bar{\nabla})$. Finally, we use this to show that if a Jacobi field of the lift of a SODE solution is a certain lift, then it can be extended to a symmetry of the SODE.

1. Introduction

In a recent series of papers, Martínez, Cariñena and Sarlet [11–13] studied the algebra of the derivations of scalar- and vector-valued forms along the tangent bundle projection $\tau : TM \to M$, generalizing the work of Foulon [10]. This work has already yielded applications: in [13], coordinate-invariant conditions were determined for the separability of systems of coupled second-order differential equations (SODE). A more recent paper [8] has shown that Douglas's rather inobvious classification of SODE, used in his solution of the inverse problem of Lagrangian mechanics [9], can be expressed quite clearly and economically using the calculus of Martínez *et al.* However, attempting to generalize Douglas's solution of the inverse problem to more than two dimensions leads to the calculation of algebraic conditions referred to by Douglas [9] as *alternants*. In order to recognize which of the conditions determined by the alternants are redundant, one requires a list of identities among the various tensor fields defined by Martínez *et al.*

This last mentioned requirement was the stimulus for this paper. While the formalism defined in [11–13] is compact and has many computational advantages (as evidenced by the applications mentioned), it is not clear from the existing presentations how one might produce such a list in a methodical fashion. This leaves open the risk that some of the conditions produced for a given problem may in fact be mere identities resulting from the geometry, so one is obliged to check by painful coordinate calculations if this is so.

The main purpose of this paper is to show that, while the calculus of objects defined along the tangent bundle projection appears at first to be unfamiliar, it can be interpreted as a more economical statement of very familiar Riemannian geometry. Using this fact, we can construct identities for use in applications such as those noted above. Further, by making precise the relationship between the generalized Jacobi equation, defined by Foulon

[10] and used in [13, 8], and the standard Jacobi equation for Riemannian geodesics, we are able to prove a stronger version of the theorem of Martínez *et al* concerning symmetries of the system of SODE.

The notion of a dynamical covariant derivative associated with an arbitrary SODE, which determines a nonlinear connection on TM, was defined in [12, 13]. The idea of nonlinear connections for SODE has been studied by Crampin [3–5] and by Morandi *et al* [14] amongst others, although, in the restricted case of SODE with force terms homogeneous of degree two in the velocities, it originates with Yano and Ishihara [17]. Generalized Bianchi identities for nonlinear connections were studied by Crampin in [5], but that paper does not show how to incorporate the other covariant derivations defined in the papers by Martínez *et al*, for example, D_*^H and D_*^V in the notation of [13]. Nor is it clear how the several curvature-like objects defined along τ in [13] should be interpreted in comparison with the familiar torsion and Riemann curvature tensors.

In this article, we shall demonstrate that by returning to the larger manifold $J^1(\mathbf{R}, M)$, the first jet bundle of smooth maps $\mathbf{R} \to M$, we obtain the objects defined in [13] as components of the conventionally defined torsion and Riemann curvature of a *linear* connection on $TJ^1(\mathbf{R}, M)$. Moreover, we calculate the full set of Bianchi identities for this connection, among which are found the various identities given in [13].

Finally, the dynamical covariant derivative, defined in [12], is written in terms of the Jacobi endomorphism Φ. This object is named for its role in the Killing equation for symmetries of a SODE field, which bears a strong formal resemblance to the equation of geodesic deviation familiar from general relativity. In that setting, the field describing the separation of neighbouring geodesics is called the Jacobi field and, hence, the name for Φ. It will be shown that this equation, in fact, follows directly from the equation of geodesic deviation on $J^1(\mathbf{R}, M)$, so that a symmetry of an SODE is, in fact, a Jacobi field.

Section 2 contains a brief outline of the material in [11–13] in order to fix notation. The notation will be that of [13], the introductory section of which provides the most compact description of the machinery involved. In section 3, we define the linear connection on $TJ^1(\mathbf{R}, M)$ in terms of the derivations defined in section 2, then calculate the torsion and curvature. Section 4 goes on to calculate the two sets of Bianchi identities. Discussion of the Jacobi endomorphism and the relationship between symmetries of the SODE and Jacobi fields on $J^1(\mathbf{R}, M)$ is in section 5, with concluding remarks in section 6.

2. The Martínez–Cariñena–Sarlet calculus

In this section, we give only a brief outline of the calculus developed in [11, 12], following the simplified notation of [13].

Take M to be a real n-dimensional C^∞ manifold. TM is the tangent bundle of M and τ is the bundle projection $\tau : TM \to M$. Then, a vector field V along τ is a map (not necessarily linear) $V : TM \to TM$ such that if $u \in T_xM$ then $V(u) \in T_xM$. That is to say that V preserves the fibres of TM but not necessarily its vector-space structure. Similarly, a 1-form α along τ is a fibre-preserving map $\alpha : TM \to T^*M$, so that $\alpha(u) \in T_x^*M$. More general tensorial objects along τ are defined via the appropriate tensor products. In natural coordinates, (x^i, u^i) for (an open subset of) TM, V and α are written:

$$V(x, u) = V^i(x, u)\frac{\partial}{\partial x^i} \qquad \alpha(x, u) = \alpha_i(x, u)\,\mathrm{d}x^i .$$

The identity map $I_\tau : TM \to TM$ is a distinguished vector field along τ, with coordinate expression $I_\tau = u^i \frac{\partial}{\partial x^i}$.

If E is a C^∞ manifold, let $\chi(E)$ denote the module of vector fields on E over the ring $C^\infty(E)$ (since all the results in this paper are local, we could as well work in a local trivialization so that the module is free). The advantage of defining vector fields along τ is that we are free to choose appropriate lifts $TM \to TTM$ which define a basis for TTU (U open in M) via lifts of a local basis for $\chi(\tau)$, the vector fields along τ. Of course the degree of advantage depends on the appropriateness of the selected lifts to the problem at hand: the formalism is really a general method for using convenient bases.

An obvious choice of lift is the vertical lift. The vertical subspace $\mathcal{V}(TM)$ of $\chi(TM)$ is defined to be the kernel of the projection $\tau_* : \chi(TM) \to \chi(M)$ in the usual way (maps between vector bundles induce maps between the modules of sections of those bundles: no distinction will be made between these). Now let $Y \in \chi(\tau)$. Then, for each $p = (x, u) \in TM$, the curve

$$t \mapsto (x, u + tY(p)) \qquad t \in (-1, 1)$$

is contained in the fibre $T_{\tau(p)}M$, so we define the vertical lift to be

$$Y^{\mathrm{V}} = \frac{\mathrm{d}}{\mathrm{d}t}(x, u + tY(p))|_{t=0}.$$

In natural coordinates (x, u), if $Y = Y^i \frac{\partial}{\partial x^i}$, then $Y^{\mathrm{V}} = Y^i \frac{\partial}{\partial u^i}$.

Notation. From this point on, I will write $X_j \equiv \frac{\partial}{\partial x^j} \in \chi(M)$ in the interests of clarity and typographical sanity. Consequently, $X_j^{\mathrm{V}} \equiv \frac{\partial}{\partial u^j}$. Abusing the notation slightly, X_j will also denote the coordinate vector field $\frac{\partial}{\partial x^j} \in \chi(TM)$ with respect to the local natural coordinates $\{x^j, u^j\}$ so that $X_j \equiv \tau_* X_j$. As a vector field on M, it is also a *basic* vector field along τ, which is to say it is defined along τ by composition with the projection. Hence, I also write $X_j \equiv X_j \circ \tau$.

In order to motivate the choice of complementary, or horizontal, subspace of TTM, first recall the definition of the *vertical endomorphism* $S : \chi(TM) \to \chi(TM)$. A coordinate-free definition is available in Morandi *et al* [14], but in local coordinates

$$S = X_j^{\mathrm{V}} \otimes \tau^* \,\mathrm{d}x^j.$$

Note that $\mathrm{im}(S) = \ker(S) = \mathcal{V}(TM)$, yielding the short exact sequence

$$0 \longrightarrow \mathcal{V}(TM) \xrightarrow{\ i\ } \chi(TM) \xrightarrow{\ S\ } \mathcal{V}(TM) \longrightarrow 0. \tag{1}$$

The map i is the inclusion map. The chosen horizontal subspace will be isomorphic to $\mathcal{V}(TM)$ via S and so split this sequence: the question is how to make the split in a meaningful way. Since the purpose of the formalism is to simplify calculations involving SODE, the associated SODE field should ideally play a role.

Let

$$\ddot{x}^j = f^j(x, u) \qquad j = 1, \ldots, n \tag{2}$$

be a system of SODE. Then the solution curves of (2) in M are precisely the projections by τ of the integral curves of the SODE field

$$\Gamma = u^j X_j + f^j(x, u) X_j^{\mathrm{V}}$$

in TM. The Lie derivative $\mathcal{L}_\Gamma S$ can be shown to have eigenvalues 1 and -1, the eigenspace for 1 being $\mathcal{V}(TM)$ (see [14]). The eigenspace for -1 is taken to be the horizontal subspace and $\chi(TM) = \mathcal{V}(TM) \oplus \mathcal{H}(TM)$ is the required splitting of (1). The horizontal lift $Y^{\mathrm{H}} \in \mathcal{H}(TM)$ of $Y \in \chi(M)$ is the unique element $Z \in \mathcal{H}(TM)$ such that $\tau_* Z = Y$. Then, since $\chi(M)$ generates $\chi(\tau)$ over $C^\infty(TM)$, the definition of the horizontal lift extends uniquely by linearity to $\chi(\tau)$. In coordinates, $X_j^{\mathrm{H}} = X_j - \Gamma_j^k X_k^{\mathrm{V}}$, where I have used $\Gamma_j^k = -\frac{1}{2} X_j^{\mathrm{V}}(f^k)$.

Notation. We will subsequently use the notation $\Gamma^m_{jk\ldots l} \equiv X_l^{\mathrm{V}} \ldots X_k^{\mathrm{V}}(\Gamma_j^m)$.

The lifts $Y \to Y^{\mathrm{V}}$ and $Y \to Y^{\mathrm{H}}$ are the 'appropriate' lifts to use in studying the SODE field Γ, as the results of [13, 2] demonstrate, so we take, as a (possibly local) basis for $\chi(TM)$, the lifted natural basis for $\chi(M)$ and, hence, for $\chi(\tau)$; $\{X_j^{\mathrm{V}}, X_j^{\mathrm{H}}\}_{j=1}^n$.

The derivations D_*^{V} and D_*^{H} on $\chi(\tau)$ are defined by

$$[X^{\mathrm{H}}, Y^{\mathrm{V}}] = \{D_X^{\mathrm{H}} Y\}^{\mathrm{V}} - \{D_Y^{\mathrm{V}} X\}^{\mathrm{H}}.$$

These satisfy a Leibnitz rule if the action of D_X^{V} and D_X^{H} on $C^\infty(TM)$ is given by

$$D_X^{\mathrm{V}}(g) = X^{\mathrm{V}}(g) \qquad D_X^{\mathrm{H}}(g) = X^{\mathrm{H}}(g).$$

They are covariant in that they are $C^\infty(TM)$ linear in the subscript argument. In coordinates, if $Y = Y^k X_k$, then

$$D_{X_j}^{\mathrm{H}} Y = (X_j^{\mathrm{H}}(Y^k) + Y^l \Gamma^k_{jl}) X_k \qquad D_{X_j}^{\mathrm{V}} Y = X_j^{\mathrm{V}}(Y^k) X_k.$$

The action of D_*^{H} and D_*^{V} on forms and other tensors along τ is then defined by duality. The exterior derivations d^{H} and d^{V} are defined by applying D^{H} or D^{V}, respectively, and then anti-symmetrizing the arguments in the usual manner of defining a covariant exterior derivative.

Some tensorial objects R, Rie and θ can now be defined by

$$[X^{\mathrm{H}}, Y^{\mathrm{H}}] = \{[X, Y]_{\mathrm{H}}\}^{\mathrm{H}} + \{R(X, Y)\}^{\mathrm{V}}$$

where

$$[X, Y]_{\mathrm{H}} = D_X^{\mathrm{H}} Y - D_Y^{\mathrm{H}} X$$

$$[D_X^{\mathrm{V}}, D_Y^{\mathrm{H}}] Z = D^{\mathrm{H}}_{D_X^{\mathrm{V}} Y} Z - D^{\mathrm{V}}_{D_Y^{\mathrm{H}} X} Z + \theta(X, Y) Z$$

$$[D_X^{\mathrm{H}}, D_Y^{\mathrm{H}}] Z = D^{\mathrm{H}}_{[X,Y]_{\mathrm{H}}} Z + D^{\mathrm{V}}_{R(Y,X)} Z + \mathrm{Rie}(X, Y) Z.$$

R is a vector-valued 2-form, Rie is a (1, 1)-tensor-valued 2-form and θ can be thought of as a (1, 1)-tensor-valued symmetric bilinear form. In coordinates,

$$R(X_j, X_k) = (-X_j^{\mathrm{H}}(\Gamma_k^m) + X_k^{\mathrm{H}}(\Gamma_j^m)) X_m$$

$$\theta(X_j, X_k) X_l = \Gamma^m_{jkl} X_m$$

$$\mathrm{Rie}(X_j, X_k) X_l = (X_j^{\mathrm{H}}(\Gamma^m_{kl}) - X_k^{\mathrm{H}}(\Gamma^m_{jl}) + \Gamma^m_{jr}\Gamma^r_{kl} - \Gamma^m_{kr}\Gamma^r_{jl}) X_m.$$

It can be shown from the coordinate expression that

$$\mathrm{Rie}(X, Y)Z = -[D_Z^{\mathrm{V}} R](X, Y).$$

The *tension* $\mathbf{t}$ is a (1, 1)-tensor field defined by $\mathbf{t} = -D^{\mathrm{H}} I_\tau$, or in coordinates by

$$\mathbf{t}(X_j) = (\Gamma_j^m - \Gamma_{jk}^m u^k) X_m.$$

If $\mathbf{t} = 0$, the connection Γ_j^m is linear and Rie is its Riemann curvature tensor.

Closely associated with the SODE field are the operators ∇ and Φ, defined by

$$[\Gamma, Y^{\mathrm{H}}] = \{\nabla Y\}^{\mathrm{H}} + \{\Phi(Y)\}^{\mathrm{V}}.$$

When evaluated on a function $g \in C^\infty(TM)$, $\nabla g = \Gamma(g)$. It is also true that

$$[\Gamma, Y^{\mathrm{V}}] = -Y^{\mathrm{H}} + \{\nabla Y\}^{\mathrm{V}}.$$

In coordinates

$$\nabla Y = (\Gamma(Y^j) + Y^k \Gamma_k^j) X_j$$

$$\Phi(X_j) = (-X_j(f^m) - \Gamma_k^m \Gamma_j^k - \Gamma(\Gamma_j^m)) X_m.$$

If Φ is differentiated as a vector-valued 1-form, we can obtain the identities

$$d^{\mathrm{V}}\Phi = 3R \qquad d^{\mathrm{H}}\Phi = \nabla R.$$

It can also be shown that $\Phi(Y) = R(I_\tau, Y) - D_Y^{\mathrm{H}} \nabla I_\tau$.

The (1, 1)-tensor field Φ is commonly called the Jacobi endomorphism (see, for example, [12]) for reasons that will be explored in some detail in section 5.

3. The linear connection on evolution space

In this section, we show how the various covariant derivatives defined on $\chi(\tau)$ in the previous section can be defined via components of a linear connection on E, the tangent bundle of evolution space. Evolution space is a common name for the first jet bundle of smooth maps $\mathbf{R} \to M$, $J^1(\mathbf{R}, M)$. We follow Crampin, Prince and Thompson [7] in the following discussion of evolution space.

A point of $E = J^1(\mathbf{R}, M)$ is an equivalence class of smooth curves in M. We can associate each curve $\sigma : \mathbf{R} \to M$ with graph $t \to (t, \sigma(t))$, a curve in $\mathbf{R} \times M$. At each fixed $t \in \mathbf{R}$, we define the usual jet-bundle equivalence relation on curves defined on a neighbourhood of t: ρ and σ are defined to be equivalent if $\rho(t) = \sigma(t)$ and $\dot\rho(t) = \dot\sigma(t)$ where $\dot\sigma(t)$ is the tangent vector to σ at t. The equivalence class of a curve σ is referred to as its 1-jet, $j_t^1(\sigma) \in J^1(\mathbf{R}, M)$. It is clear that for each $t \in \mathbf{R}$, the set of equivalence classes can be identified with TM, so $J^1(\mathbf{R}, M)$ can be identified with $\mathbf{R} \times TM$. We shall use local coordinates (t, x, u) for $\mathbf{R} \times TM = J^1(\mathbf{R}, M)$.

The projection $\tau : TM \to M$ pulls back to a projection $\mathbf{R} \times TM \to \mathbf{R} \times M$ or, equivalently, $J^1(\mathbf{R}, M) \to J^0(\mathbf{R}, M)$. By a slight abuse of notation this projection will also be written as τ. Given a curve $t \to (t, \sigma(t))$ in $\mathbf{R} \times M$, we can define its *complete lift* to $J^1(\mathbf{R}, M)$ to be $t \to (t, \sigma(t), \dot\sigma(t))$. Note that it is a cross section of τ. On the other hand, if the graph $t \to (t, \sigma(t), u(t))$ of an arbitrary curve in TM is the lift of any curve in the base space, it must be the lift of its projection by τ. This is the case only if

$$u(t) = \dot{\sigma}(t) \qquad \forall\, t \in \text{domain}(\sigma).$$

This condition is commonly expressed in terms of the n-contact 1-forms $\{\theta^a\}$ which have coordinate representation

$$\theta^a = \mathrm{d}x^a - u^a\, \mathrm{d}t.$$

Now, a section ζ of $J^1(\mathbf{R}, M) \to \mathbf{R}$ is the lift of its projection to $J^0(\mathbf{R}, M)$ iff

$$\zeta^*\theta^a = 0 \qquad a = 1, \ldots, m.$$

Also, since ζ is a section, it has the form

$$\zeta : t \longrightarrow (t, \sigma(t), u(t))$$

implying that $\zeta^*\, \mathrm{d}t = \mathrm{d}t$ (and thereby excusing our notation).

It follows from the above that if Γ is a vector field on $E = J^1(\mathbf{R}, M)$, then its integral curves are lifts of curves on $\mathbf{R} \times M$ iff it satisfies the conditions

$$\Gamma \lrcorner\, \mathrm{d}t = 1 \qquad \Gamma \lrcorner\, \theta^a = 0 \qquad a = 1, \ldots, m.$$

Any such vector field is called an SODE field on E: we have modified the definition of an SODE field on TM given in section 2 by adding $\frac{\partial}{\partial t}$. In coordinates, we now have

$$\Gamma = \frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^j} + f^j(t, x, u) \frac{\partial}{\partial u^j}.$$

Any SODE field on TM can be defined on E simply by treating t as a parameter and adding the $\frac{\partial}{\partial t}$ term. On the other hand, working on E allows us to deal with non-autonomous differential equations. We extend the definitions of vector fields and forms along $\tau : TM \to M$ to the time-dependent case $\tau : \mathbf{R} \times TM \to \mathbf{R} \times M$ via a minimalist procedure: we simply transfer the machinery of section 2 to each constant time slice $\{t\} \times TM$. Thus, an element $X \in \chi(\tau)$ is a fibre-preserving map $X : E \to TM$ and $X(t) = 0$. The definitions of X^{V} and X^{H} are then exactly as before within each fibre, so that also $X^{\mathrm{V}}(t) = X^{\mathrm{H}}(t) = 0$. Hence, we retain $d^{\mathrm{V}} \circ d^{\mathrm{V}} = 0$ and other nice properties of the autonomous situation. Note that this is *not* the philosophy of, for example, [7], where basic forms $\mathrm{d}x^j$ are systematically replaced with the corresponding contact forms $\theta^j = \mathrm{d}x^j - u^j\, \mathrm{d}t$. For a more comprehensive survey of the range of alternatives available when moving to the time-dependent case, see Sarlet, Vandecasteele, Cantrijn and Martínez [15].

In fact, since we only evaluate the various tensors defined in the autonomous case on vectors X^{V} and X^{H}, defined to be tangent to $\{t\} \times TM \subset E$, the choice between $\mathrm{d}x^j$ and θ^j is immaterial. However, we will stay with the minimal option for the sake of definiteness and simplicity.

Given the foregoing discussion, the (local) basis $\{X_j^{\mathrm{V}}, X_j^{\mathrm{H}}\}_{j=1}^n$ for $\chi(TM)$ defined in section 2 extends trivially to a (local) basis $\{\Gamma, X_j^{\mathrm{V}}, X_j^{\mathrm{H}}\}_{j=1}^n$ for $\chi(E)$.

Notation. If $X \in \chi(\tau)$, we will use the notation $X^{\mathrm{V|H}}$ as shorthand for 'X^{V}, respectively X^{H}'.

To define a connection on E, we make the following ansatz (with $X, Y \in \chi(\tau)$):

$$\begin{aligned} \tilde{\nabla}_{X^{\mathrm{V}}} Y^{\mathrm{V|H}} &= \{D^{\mathrm{V}}_X Y\}^{\mathrm{V|H}} & \qquad \tilde{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{V|H}} &= \{D^{\mathrm{H}}_X Y\}^{\mathrm{V|H}} \\ \tilde{\nabla}_{\Gamma} Y^{\mathrm{V|H}} &= \{\nabla Y\}^{\mathrm{V|H}} & \qquad \tilde{\nabla}_{\Gamma}\Gamma &= 0. \end{aligned} \tag{3}$$

We also impose that it be a *linear* connection: if $Z, W \in \chi(E)$ and $g \in C^{\infty}(E)$, then

$$\tilde{\nabla}_{gZ} W = g\tilde{\nabla}_Z W \qquad \tilde{\nabla}_Z(gW) = Z(g)W + g\tilde{\nabla}_Z W.$$

It can now be seen why we needed to introduce E in place of TM: as defined in section 2, $\{\Gamma, X^{\mathrm{V}}_j, X^{\mathrm{H}}_j\}^n_{j=1}$ is linearly dependent, which would preclude defining a linear connection with the above properties. Note that the list (3) above does not fully define $\tilde{\nabla}$. We have left $\tilde{\nabla}_{X^{\mathrm{V}}}\Gamma$ and $\tilde{\nabla}_{X^{\mathrm{H}}}\Gamma$ to be determined by a condition on the torsion of $\tilde{\nabla}$.

As a first step to the torsion, we recall the following commutation relations:

$$\begin{aligned} &[X^{\mathrm{V}}_j, X^{\mathrm{V}}_k] = 0 \\ &[X^{\mathrm{V}}_j, X^{\mathrm{H}}_k] = -\Gamma^m_{jk} X^{\mathrm{V}}_m = -D^{\mathrm{H}}_{X_k} X_j \\ &[X^{\mathrm{H}}_j, X^{\mathrm{H}}_k] = (-X^{\mathrm{H}}_j(\Gamma^m_k) + X^{\mathrm{H}}_k(\Gamma^m_j))X^{\mathrm{V}}_m = \{R(X_j, X_k)\}^{\mathrm{V}} \\ &[\Gamma, X^{\mathrm{V}}_j] = -X^{\mathrm{H}}_j + \Gamma^k_j X^{\mathrm{V}}_k \\ &[\Gamma, X^{\mathrm{H}}_j] = \Gamma^k_j X^{\mathrm{H}}_k + \{\Phi(X_j)\}^{\mathrm{V}} = \{\nabla X_j\}^{\mathrm{H}} + \{\Phi(X_j)\}^{\mathrm{V}}. \end{aligned}$$

Then, calculating the torsion from the usual definition

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \qquad X, Y \in \chi(E)$$

we find

$$\begin{aligned} &\tilde{T}(X^{\mathrm{V}}_j, X^{\mathrm{V|H}}_k) = 0 \\ &\tilde{T}(X^{\mathrm{H}}_j, X^{\mathrm{H}}_k) = -\{R(X_j, X_k)\}^{\mathrm{V}} \\ &\tilde{T}(\Gamma, X^{\mathrm{V}}_j) = -\tilde{\nabla}_{X^{\mathrm{V}}_j}\Gamma + X^{\mathrm{H}}_j \\ &\tilde{T}(\Gamma, X^{\mathrm{H}}_j) = -\tilde{\nabla}_{X^{\mathrm{H}}_j}\Gamma - \{\Phi(X_j)\}^{\mathrm{V}}. \end{aligned}$$

We now complete our definition of $\tilde{\nabla}$ by requiring

$$\tilde{T}(\Gamma, Z) = 0 \qquad \forall\ Z \in \chi(E).$$

As a result of this choice, the only non-zero components of $\tilde{T}$ are given by

$$\tilde{T}(X^{\mathrm{H}}_j, X^{\mathrm{H}}_k) = -\{R(X_j, X_k)\}^{\mathrm{V}}$$

and the components of the connection are given by

$$\begin{aligned} &\tilde{\nabla}_{X^{\mathrm{V}}_j} X^{\mathrm{V|H}}_k = 0 & \qquad &\tilde{\nabla}_{X^{\mathrm{H}}_j} X^{\mathrm{V|H}}_k = \Gamma^m_{jk} X^{\mathrm{V|H}}_m \\ &\tilde{\nabla}_{\Gamma} X^{\mathrm{V}}_j = \Gamma^k_j X^{\mathrm{V}}_k & \qquad &\tilde{\nabla}_{\Gamma} X^{\mathrm{H}}_j = \Gamma^k_j X^{\mathrm{H}}_k \\ &\tilde{\nabla}_{X^{\mathrm{V}}_j}\Gamma = X^{\mathrm{H}}_j & \qquad &\tilde{\nabla}_{X^{\mathrm{H}}_j}\Gamma = -\{\Phi(X_j)\}^{\mathrm{V}}. \end{aligned}$$

From the above and the definition of Γ, we have that

$$\tilde{\nabla}_\Gamma = \tilde{\nabla}_{\frac{\partial}{\partial t}} + u^j \tilde{\nabla}_{X_j^{\mathrm{H}}} + (u^k \Gamma_k^j + f^j)\tilde{\nabla}_{X_j^{\mathrm{V}}}$$

and, hence, we can show that

$$\tilde{\nabla}_{\frac{\partial}{\partial t}} X_j^{\mathrm{V|H}} = \{\mathbf{t}(X_j)\}^{\mathrm{V|H}}$$

where $\mathbf{t}$ is the tension defined in section 2. This demonstrates that the need to replace TM with E arises precisely from the nonlinearity of the connection Γ_j^k on TM, as measured by $\mathbf{t}$.

Since $\tilde{\nabla}_\Gamma \Gamma = 0$, we have also that

$$\tilde{\nabla}_{\frac{\partial}{\partial t}} \Gamma = u^j\{\Phi(X_j)\}^{\mathrm{V}} - (u^k \Gamma_k^j + f^j) X_j^{\mathrm{V}}.$$

In the case of an autonomous SODE (i.e. $\partial f^j/\partial t = 0$), this reduces to

$$\tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = u^j\{\mathbf{t}(X_j)\}^{\mathrm{H}} + (u^k \Gamma_k^j + f^j)\{\mathbf{t}(X_j)\}^{\mathrm{V}}.$$

From this point, it is straightforward to calculate the components of the Riemann curvature of $\tilde{\nabla}$, defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \qquad X, Y, Z \in \chi(E).$$

The components are as follows

$$\tilde{R}(X_j^{\mathrm{V}}, X_k^{\mathrm{V}}) X_l^{\mathrm{V|H}} = 0$$

$$\tilde{R}(X_j^{\mathrm{V}}, X_k^{\mathrm{V}}) \Gamma = 0$$

$$\tilde{R}(X_j^{\mathrm{V}}, X_k^{\mathrm{H}}) X_l^{\mathrm{V|H}} = \{\theta(X_j, X_k) X_l\}^{\mathrm{V|H}}$$

$$\tilde{R}(X_j^{\mathrm{V}}, X_k^{\mathrm{H}}) \Gamma = -\{D^{\mathrm{V}}_{X_j} \Phi(X_k)\}^{\mathrm{V}}$$

$$\tilde{R}(X_j^{\mathrm{H}}, X_k^{\mathrm{H}}) X_l^{\mathrm{V|H}} = \{\mathrm{Rie}(X_j, X_k) X_l\}^{\mathrm{V|H}}$$

$$\tilde{R}(X_j^{\mathrm{H}}, X_k^{\mathrm{H}}) \Gamma = -\{d^{\mathrm{H}}\Phi(X_j, X_k)\}^{\mathrm{V}} - \{R(X_j, X_k)\}^{\mathrm{H}}$$

$$\tilde{R}(\Gamma, X_j^{\mathrm{V}}) X_k^{\mathrm{V|H}} = 0$$

$$\tilde{R}(\Gamma, X_j^{\mathrm{V}}) \Gamma = -\{\Phi(X_j)\}^{\mathrm{V}}$$

$$\tilde{R}(\Gamma, X_j^{\mathrm{H}}) X_k^{\mathrm{V|H}} = -\{D^{\mathrm{V}}_{X_k} \Phi(X_j) + R(X_j, X_k)\}^{\mathrm{V|H}}$$

$$\tilde{R}(\Gamma, X_j^{\mathrm{H}}) \Gamma = -\{\nabla\Phi(X_j)\}^{\mathrm{V}} - \{\Phi(X_j)\}^{\mathrm{H}}.$$

4. Bianchi identities

Having calculated the Riemann curvature and the torsion in the previous section, it is now only a matter of tedious calculation to derive the Bianchi identities. The first set of these comes from the identity

$$\sum_{\text{cyclic}} \tilde{R}(X,Y)Z = \sum_{\text{cyclic}} (\tilde{T}(\tilde{T}(X,Y),Z) + [\tilde{\nabla}_X \tilde{T}](Y,Z)) \qquad \forall\, X,Y,Z \in \chi(E).$$

Since we know that

$$\tilde{T}(X,Y) \in \mathcal{V}(E) \qquad \forall\, X,Y \in \chi(E)$$

and that $\tilde{T}$ vanishes if either of its arguments are vertical, the first term on the right-hand side of the identity vanishes.

Notation. If A is a tensorial object on E (respectively, along τ) and $W,\ldots,Z \in \chi(E)$ (respectively, $\chi(\tau)$), then we will often write $\tilde{\nabla}A(W,X,\ldots,Z)$ in place of $[\tilde{\nabla}_W A](X,\ldots,Z)$ (respectively, $D^{\mathrm{V|H}}A(W,X,\ldots,Z)$ in place of $D_W^{\mathrm{V|H}}A(X,\ldots,Z)$), to avoid confusion in the case of composed derivatives.

It can be readily verified that the only non-zero components of $\tilde{\nabla}\tilde{T}$ are

$$\tilde{\nabla}\tilde{T}(X_j^{\mathrm{V}}, X_k^{\mathrm{H}}, X_l^{\mathrm{H}}) = -\{D^{\mathrm{V}}R(X_j,X_k,X_l)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{T}(X_j^{\mathrm{V}}, X_k^{\mathrm{H}}, \Gamma) = -\{R(X_j,X_k)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{T}(X_j^{\mathrm{H}}, X_k^{\mathrm{H}}, X_l^{\mathrm{H}}) = -\{D^{\mathrm{H}}R(X_j,X_k,X_l)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{T}(\Gamma, X_j^{\mathrm{H}}, X_k^{\mathrm{H}}) = -\{\nabla R(X_j,X_k)\}^{\mathrm{V}}.$$

Using these, we obtain the first set of Bianchi identities

$$\theta(X_j,X_k)X_l - \theta(X_k,X_l)X_j = 0 \tag{4}$$

$$\mathrm{Rie}(X_j,X_k)X_l + D^{\mathrm{V}}R(X_l,X_j,X_k) = 0 \tag{5}$$

$$\sum_{\text{cyclic}} \mathrm{Rie}(X_j,X_k)X_l = 0 \tag{6}$$

$$d^{\mathrm{H}}R(X_j,X_k,X_l) = 0 \tag{7}$$

$$d^{\mathrm{H}}\Phi - \nabla R = 0 \tag{8}$$

$$d^{\mathrm{V}}\Phi - 3R = 0. \tag{9}$$

The mysterious factor of three in (9) comes about as a result of both the anti-symmetrization of the exterior derivative, giving a factor of two, and the contribution from the torsion. The identities (5), (8) and (9) are all noted in [13].

The second set of Bianchi identities result from the identity

$$\sum_{\text{cyclic}} \tilde{\nabla}\tilde{R}(X,Y,Z) + \tilde{R}(\tilde{T}(X,Y),Z) = 0 \qquad \forall\, X,Y,Z \in \chi(E). \tag{10}$$

We record the following intermediate results in the hope of saving others from tedious calculation:

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{V}}, X_k^{\mathrm{V}})X_l^{\mathrm{V|H}} = 0$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{V}}, X_k^{\mathrm{V}})\Gamma = 0$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{V}}, X_k^{\mathrm{H}})X_l^{\mathrm{V|H}} = \{D^{\mathrm{V}}\theta(X_i, X_j, X_k)X_l\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{V}}, X_k^{\mathrm{H}})\Gamma = -\{(D^{\mathrm{V}})^2\Phi(X_i, X_j, X_k)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{H}}, X_k^{\mathrm{H}})X_l^{\mathrm{V|H}} = \{D^{\mathrm{V}}\,\mathrm{Rie}(X_i, X_j, X_k)X_l\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, X_j^{\mathrm{H}}, X_k^{\mathrm{H}})\Gamma = -\{D^{\mathrm{V}}_{X_i}d^{\mathrm{H}}\Phi(X_j, X_k)\}^{\mathrm{V}} - \{D^{\mathrm{V}}R(X_i, X_j, X_k) + \mathrm{Rie}(X_j, X_k)X_i\}^{\mathrm{H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, \Gamma, X_j^{\mathrm{V}})X_k^{\mathrm{V|H}} = \{\theta(X_i, X_j)X_k\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, \Gamma, X_j^{\mathrm{V}})\Gamma = -\{d^{\mathrm{V}}\Phi(X_i, X_j)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, \Gamma, X_j^{\mathrm{H}},)X_k^{\mathrm{V|H}} = -\{D^{\mathrm{V}}D^{\mathrm{V}}\Phi(X_i, X_k, X_j) + D^{\mathrm{V}}R(X_i, X_j, X_k) + \mathrm{Rie}(X_i, X_j)X_k\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{V}}, \Gamma, X_j^{\mathrm{H}})\Gamma = \{-D^{\mathrm{V}}\nabla\Phi(X_i, X_j) + d^{\mathrm{H}}\Phi(X_i, X_j)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{V}}, X_k^{\mathrm{V}})X_l^{\mathrm{V|H}} = 0$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{V}}, X_k^{\mathrm{V}})\Gamma = 0$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{V}}, X_k^{\mathrm{H}})X_l^{\mathrm{V|H}} = \{D^{\mathrm{H}}\theta(X_i, X_j, X_k)X_l\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{V}}, X_k^{\mathrm{H}})\Gamma = -\{D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_i, X_j, X_k) + \theta(X_j, X_k)\Phi(X_l)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{H}}, X_k^{\mathrm{H}})X_l^{\mathrm{V|H}} = \{D^{\mathrm{H}}\,\mathrm{Rie}(X_i, X_j, X_k)X_l\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, X_j^{\mathrm{H}}, X_k^{\mathrm{H}})\Gamma = \{\mathrm{Rie}(X_j, X_k)\Phi(X_i) - D^{\mathrm{H}}d^{\mathrm{H}}\Phi(X_i, X_j, X_k)\}^{\mathrm{V}} - \{D^{\mathrm{H}}R(X_i, X_j, X_k)\}^{\mathrm{H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, \Gamma, X_j^{\mathrm{V}})X_k^{\mathrm{V|H}} = 0$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, \Gamma, X_j^{\mathrm{V}})\Gamma = -\{D^{\mathrm{H}}\Phi(X_i, X_j)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, \Gamma, X_j^{\mathrm{H}})X_k^{\mathrm{V|H}} = -\{D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_i, X_k, X_j) + D^{\mathrm{H}}R(X_i, X_j, X_k)\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(X_i^{\mathrm{H}}, \Gamma, X_j^{\mathrm{H}})\Gamma = -\{R(X_j, \Phi(X_i)) + D^{\mathrm{H}}\nabla\Phi(X_i, X_j) + 2D^{\mathrm{V}}\Phi(\Phi(X_i), X_j)\}^{\mathrm{V}} - \{D^{\mathrm{H}}\Phi(X_i, X_j)\}^{\mathrm{H}}$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{V}}, X_j^{\mathrm{V}})X_k^{\mathrm{V|H}} = 0$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{V}}, X_j^{\mathrm{V}})\Gamma = 0$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{V}}, X_j^{\mathrm{H}})X_k^{\mathrm{V|H}} = \{\nabla\theta(X_i, X_j)X_k\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{V}}, X_j^{\mathrm{H}})\Gamma = -\{\nabla D^{\mathrm{V}}\Phi(X_i, X_j)\}^{\mathrm{V}}$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{H}}, X_j^{\mathrm{H}})X_k^{\mathrm{V|H}} = \{\nabla\,\mathrm{Rie}(X_i, X_j)X_k\}^{\mathrm{V|H}}$$

$$\tilde{\nabla}\tilde{R}(\Gamma, X_i^{\mathrm{H}}, X_j^{\mathrm{H}})\Gamma = -\{\nabla d^{\mathrm{H}}\Phi(X_i, X_j)\}^{\mathrm{V}} - \{\nabla R(X_i, X_j)\}^{\mathrm{H}}.$$

Substituting the above expressions into (10) yields the following identities (those which appeared in the first set are omitted):

$$D^{\mathrm{V}}\theta(X_i, X_j, X_k) - D^{\mathrm{V}}\theta(X_j, X_i, X_k) = 0$$

$$D^{\mathrm{V}}\,\mathrm{Rie}(X_i, X_j, X_k) - D^{\mathrm{H}}\theta(X_j, X_i, X_k) + D^{\mathrm{H}}\theta(X_i, X_j, X_k) = 0$$

$$D^{\mathrm{V}}d^{\mathrm{H}}\Phi(X_i, X_j, X_k) - D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_j, X_i, X_k) + D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_k, X_i, X_j)$$
$$- \theta(X_i, X_k)\Phi(X_j) + \theta(X_i, X_j)\Phi(X_k) = 0$$

$$d^{\mathrm{H}}\,\mathrm{Rie}(X_i, X_j, X_k) - \sum_{\text{cyclic}} \theta(R(X_i, X_j), X_k) = 0$$

$$\sum_{\text{cyclic}} (\mathrm{Rie}(X_i, X_j) + D^{\mathrm{V}}_{R(X_i, X_j)})\Phi(X_k) = 0$$

$$\nabla\theta(X_i, X_j)X_k + (D^{\mathrm{V}})^2\Phi(X_i, X_j, X_k) + D^{\mathrm{V}}R(X_i, X_j, X_k) + \mathrm{Rie}(X_i, X_j)X_k = 0$$

$$D^{\mathrm{V}}\nabla\Phi - \nabla D^{\mathrm{V}}\Phi - D^{\mathrm{H}}\Phi = 0$$

$$[\nabla, d^{\mathrm{H}}]\Phi(X_i, X_j) + R(X_i, \Phi(X_j)) - R(X_j, \Phi(X_i))$$
$$- 2D^{\mathrm{V}}\Phi(\Phi(X_i), X_j) + 2D^{\mathrm{V}}\Phi(\Phi(X_j), X_i) = 0$$

$$\nabla\,\mathrm{Rie}(X_i, X_j)X_k + D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_i, X_k, X_j) - D^{\mathrm{H}}D^{\mathrm{V}}\Phi(X_j, X_k, X_i) - D^{\mathrm{H}}R(X_k, X_i, X_j) = 0.$$

5. Geodesic deviation and the Jacobi endomorphism

In defining the connection $\tilde{\nabla}$ so that $\tilde{\nabla}_\Gamma\Gamma = 0$, we ensure that the integral curves of Γ are geodesics of $\tilde{\nabla}$, at least in the sense of being self-parallel (we have not specified a metric on E invariant under $\tilde{\nabla}$ and it is, in fact, possible that no such metric exists). Hence, if $\eta \in \chi(E)$ is such that $[\Gamma, \eta] = 0$ on some open set, the definition of $\tilde{R}$ gives, on a possibly smaller open set,

$$\tilde{\nabla}_\Gamma\tilde{\nabla}_\eta\Gamma = \tilde{R}(\Gamma, \eta)\Gamma$$

while choosing $\tilde{\nabla}$ so that $\tilde{T}(\Gamma, X) = 0\;\forall\, X \in \chi(E)$ implies that $\tilde{\nabla}_\Gamma\eta - \tilde{\nabla}_\eta\Gamma = 0$. Combining these two results, we see that η is a Jacobi field for an integral curve of Γ, since it satisfies the Jacobi equation or an equation of geodesic deviation:

$$\tilde{\nabla}_\Gamma^2\eta = \tilde{R}(\Gamma, \eta)\Gamma. \tag{11}$$

Note that since $\tilde{R}$ is tensorial, the Jacobi equation is *linear* along a given integral curve γ. Once γ is determined, the above equation for η can be used to determine large-scale behaviour of geodesics, such as the existence of caustics and the Milnor index theorem (see Spivak [16]). Although some steps have been taken elsewhere to extend such techniques to general SODE [6], we are not aware of an explicit Jacobi equation having been exhibited elsewhere. Nor is it immediately clear that equation (11) is the appropriate tool: while every integral curve of Γ is a geodesic, the converse is clearly false. There is one integral curve of Γ through each point p in E, whereas the geodesics through p are parametrized by the unit sphere in T_pE. We will show below that the extra freedom can be constrained by a simple condition on η.

There is a known Jacobi equation for an arbitrary SODE. Martínez, Cariñena and Sarlet [12] note that the equation

$$\nabla^2\sigma = -\Phi(\sigma) \qquad \sigma \in \chi(\tau)$$

is 'reminiscent of the concept of a Jacobi field' and describe it as a generalized Jacobi equation. In fact, it is closely related to (11).

Lemma 1. Let γ be an integral curve of Γ and let $\eta \in \chi(E)$ have decomposition $\eta = h\Gamma + \sigma^{\mathrm{H}} + \xi^{\mathrm{V}}$, $\sigma, \xi \in \chi(\tau)$ and $h \in C^\infty(E)$. Then η is a Jacobi field for γ if and only if

$$\Gamma^2(h) = 0 \qquad (\nabla^2\sigma + \Phi(\sigma))|_\gamma = 0 \qquad (\nabla^2\xi + \Phi(\xi) + \nabla\Phi(\sigma))|_\gamma = 0.$$

Proof. Substitute the decomposition of η into (11) and use the definitions

$$\tilde{R}(\Gamma, \sigma^{\mathrm{H}})\Gamma = -\{\nabla\Phi(\sigma)\}^{\mathrm{V}} - \{\Phi(\sigma)\}^{\mathrm{H}}$$

$$\tilde{R}(\Gamma, \xi^{\mathrm{V}})\Gamma = -\{\Phi(\xi)\}^{\mathrm{V}}$$

and equate the components to obtain the result. □

Since only transverse Jacobi fields are of interest, we will henceforth set $h = 0$ so that $\eta = \sigma^{\mathrm{H}} + \xi^{\mathrm{V}}$.

Lemma 2. If $\eta = \sigma^{\mathrm{H}} + (\nabla\sigma)^{\mathrm{V}}$ and $\nabla^2\sigma + \Phi(\sigma) = 0$ along γ, then η is a Jacobi field along γ.

Proof. Substituting $\xi = \nabla\sigma$ into the second condition of lemma 1

$$(\nabla^3\sigma + \Phi(\nabla\sigma) + \nabla\Phi(\sigma))|_\gamma = \nabla\left(\nabla^2\sigma + \Phi(\sigma)\right)|_\gamma = 0. \qquad \square$$

Note that the converse of lemma 2 is certainly false: there will be solutions of (11) with $\eta = \xi^{\mathrm{V}}$.

Also, note that in the notation of [11, 12]

$$J_\Gamma\sigma = \sigma^{\mathrm{H}} + (\nabla\sigma)^{\mathrm{V}}.$$

From the two lemmas, we have

Theorem 3. Let $\sigma \in \chi(\tau)$. Then $J_\Gamma\sigma$ is a Jacobi field for an integral curve γ of Γ if and only if

$$(\nabla^2\sigma + \Phi(\sigma))|_\gamma = 0.$$

Martínez *et al* have shown that $\nabla^2\sigma + \Phi(\sigma) = 0$ iff $J_\Gamma\sigma$ is a symmetry of Γ. In fact it is possible to strengthen their result:

Lemma 4. Let η be any vector field on E such that $[\Gamma, \eta] = 0$ and $\eta = \sigma^{\mathrm{H}} + \xi^{\mathrm{V}}$. Then $\eta = J_\Gamma\sigma$ and $\nabla^2\sigma + \Phi(\sigma) = 0$.

Proof. It suffices to calculate

$$[\Gamma, \eta] = \{\nabla\xi - \Phi(\sigma)\}^{\mathrm{V}} + \{\nabla\sigma - \xi\}^{\mathrm{H}} = 0.$$

This implies that

$$\eta = \sigma^{\mathrm{H}} + (\nabla\sigma)^{\mathrm{V}} = J_\Gamma\sigma \qquad \nabla^2\sigma + \Phi(\sigma) = 0. \qquad \square$$

Theorem 5. If $J_\Gamma\sigma$ is a Jacobi field for an integral curve γ of Γ, then $J_\Gamma\sigma$ is tangent to the transverse curves of a 1-parameter family of integral curves of Γ and is the restriction to γ of a symmetry of Γ.

Proof. Let p be a point on γ. With $\epsilon > 0$, let $\alpha : [-\epsilon, \epsilon] \to E$ be a curve such that $\alpha(0) = p$ and $\mathrm{d}\alpha(s)/\mathrm{d}s|_{s=0} = J_\Gamma\sigma(p)$. Then, construct the integral curve γ_s of Γ through each point $\alpha(s)$. We have constructed a 1-parameter family of integral curves containing γ, so the transverse vector field $Z = \mathrm{d}\gamma_s/\mathrm{d}s$ is a symmetry of Γ. This is equivalent to $[\Gamma, Z] = 0$, so from the lemma there exists $\rho \in \chi(\tau)$ such that $Z = J_\Gamma\rho$ and $\nabla^2\rho + \Phi(\rho) = 0$. However, from the construction

$$\rho^{\mathrm{H}}(p) + (\nabla\rho)^{\mathrm{V}}(p) = J_\Gamma\sigma(p) = \sigma^{\mathrm{H}}(p) + (\nabla\sigma)^{\mathrm{V}}(p)$$

so σ and ρ satisfy the same linear SODE along γ and have the same value and first derivative at p. Thus, $(\rho - \sigma)|_\gamma = 0$, so we have proved that $J_\Gamma\sigma$ is tangent to the transverse curves of a 1-parameter family of integral curves.

It remains to show that Z can be extended to a symmetry in a tubular neighbourhood of γ. This will be done by producing a local foliation of E by 1-parameter families of integral curves.

Denote the two-dimensional submanifold defined by the 1-parameter family of curves above by E_2. Let N be a $(2n-1)$-dimensional submanifold transverse to E_2 and to Γ with the point $E_2 \cap N$ labelled by q. We also require (without loss of generality) that no integral curve of Γ intersect N more than once. Let W be any smooth map

$$W : N \longrightarrow TE$$

which satisfies

$$W(q) = Z(q) \qquad W(a) \notin TN \qquad W(a) \neq \lambda\Gamma \qquad a \in N \qquad \lambda \in \mathbf{R}.$$

N parametrizes a $(2n-1)$-parameter family of integral curves: denote by γ_a that which passes through $a \in N$.

The vector $W(a)$ has a decomposition $W(a) = \xi^{\mathrm{H}} + \eta^{\mathrm{V}}$, so if $\rho \in \chi(\tau)$ satisfies

$$\rho(a) = \xi \qquad \nabla\rho(a) = \eta$$

then

$$W(a) = J_\gamma\rho(a).$$

The above conditions on ρ constitute initial conditions for a solution of the Jacobi equation

$$(\nabla^2\rho + \Phi(\rho))|_{\gamma_a} = 0$$

along γ_a. It follows from lemma 2 that $J_\Gamma\rho$ is a Jacobi field along γ_a, so we may use the first part of the proof to construct a 1-parameter family of integral curves through γ_a, $E_2(a)$. The transverse field in each of these families is a Jacobi field for the curves of the family. Since $\Gamma \notin TN$ and $W(a) \notin TN$, $E_2(a)$ is transverse to N for each $a \in N$.

Now suppose that $E_2(a) \cap E_2(b) \neq \emptyset$ with $a, b \in N$. Then $E_2(a) \cap E_2(b)$ must be an integral curve of Γ (as Γ is non-singular) passing through both a and b. Hence, $a = b$ and we have shown that N parametrizes a local foliation of E by the 1-parameter families $E_2(c)$. On each leaf $E_2(a)$, the transverse field Z_a satisfies $[\Gamma, Z_a] = 0$ (from the first part of the proof) and Z_a depends smoothly on a, since it is the solution of a linear SODE with smooth parameter dependence of the coefficient functions. Hence, we have a smooth extension $Z = J_\Gamma\rho$ which is a symmetry of Γ in a tubular neighbourhood of γ. □

The point of the theorem is to show that a Jacobi field for γ is not simply a variation through geodesics, which is well known [16], but is in fact a variation through those geodesics of $\tilde{\nabla}$ which are also integral curves of Γ.

The difference between theorem 5 and that proved by Martínez *et al* is that in [12], $\nabla^2\sigma + \Phi(\sigma) = 0$ is required to hold on a tubular neighbourhood of γ: here we have shown that if the condition holds only on γ there is an extension ρ of σ to a tubular neighbourhood such that ρ is a symmetry of Γ.

Proposition 6. $J_\Gamma\sigma$ is a variation of γ through integral curves of Γ iff σ is a variation of $\tau \circ \gamma$ through solution curves of the SODE in $\mathbf{R} \times M$.

Proof. The forward implication is immediate on projecting the family of integral curves. To prove the converse, note that $J_\Gamma\sigma = \sigma^j X_j + \Gamma(\sigma^j)X_j^{\mathrm{V}}$. Thus, if σ is a variation tangent to a family of solution curves, $J_\Gamma\sigma$ will be a variation through the complete lift of that family. □

Note that we have two sets of symmetries available at this point: symmetries of Γ as an SODE field on E, which is to say symmetries of the SODE, and symmetries of the geodesic equation for the connection $\tilde{\nabla}$ on E. This last equation itself defines a SODE field on TE (rather than on E) and has the form $\tilde{\nabla}_{\dot{\mu}}\dot{\mu} = 0$ with $\mu : \mathbf{R} \to E$. To summarize the relation between these two sets, we state the following corollary.

Corollary 7. If $\tilde{\nabla}$ is the linear connection on TE associated with the SODE field Γ, then the following statements are true.

(i) Every symmetry of Γ is a symmetry of $\tilde{\nabla}_{\dot{\mu}}\dot{\mu} = 0$.

(ii) There exist symmetries of $\tilde{\nabla}_{\dot{\mu}}\dot{\mu} = 0$ along the integral curves γ of Γ which are not symmetries of Γ.

Proof. Since every integral curve of Γ is a geodesic with respect to $\tilde{\nabla}$ on E, a symmetry of Γ is tangent to a family of solution curves of $\tilde{\nabla}_{\dot{\mu}}\dot{\mu} = 0$. The first part (i) then follows by applying proposition 6 one level up.

We noted after lemma 2 that there are solutions η to the Jacobi equation $\tilde{\nabla}^2_\Gamma\eta = \tilde{R}(\Gamma, \eta)\Gamma$ with $\eta = \xi^{\mathrm{V}}$ for some ξ defined along the projection $\tau\circ\gamma$ of γ to a solution curve in $\mathbf{R}\times M$.

It is a classical result (see [16]) that any Jacobi field generates a variation through geodesics and so by theorem 5 extends to a symmetry of the geodesic equation. On the other hand, η can be a symmetry of Γ only if there is a solution σ of the generalized Jacobi equation which lifts to η, i.e. $J_\Gamma\sigma = \sigma^{\mathrm{H}} + (\nabla\sigma)^{\mathrm{V}} = \eta$. Clearly, if $\eta = \xi^{\mathrm{V}}$ then there is no such σ so ξ is not a symmetry of Γ. □

Definition 1. We will say that $\sigma \in \chi(\tau)$ is a *generalized Jacobi field* along the projection $\tau \circ \gamma$ if $J_\Gamma\sigma$ is a Jacobi field along γ.

It is then natural to define conjugate points along $\tau\circ\gamma$ as points $\tau\circ\gamma(a)$, $\tau\circ\gamma(b)$ where some generalized Jacobi field σ, not identically zero, satisfies $\sigma(\gamma(a)) = \sigma(\gamma(b)) = 0$. At the conjugate points $J_\Gamma\sigma \neq 0$, otherwise $\nabla\sigma$ would vanish coincidentally with σ, implying that $\sigma \equiv 0$. In other words, at conjugate points, $J_\Gamma\sigma$ is vertical, corresponding to a variation in direction rather than position of curves in $\mathbf{R} \times M$, as one would expect.

6. Discussion

The principal aim of this paper was to show that the formal calculus of tensor fields along the tangent bundle projection, as defined by Martínez *et al*, is simply a more economical notation for ordinary tensor calculus on evolution space $E = \mathbf{R} \times TM$. By choosing an appropriate linear connection on E, all the geometric objects arising from the nonlinear connection on $\mathbf{R} \times M$ are recovered. Consequently, we are able to derive a set of identities between the objects defined in [11–13] by calculating the Bianchi identities for the curvature of the linear connection.

In section 5, it is shown that the generalized Jacobi equation for symmetries of SODE, as defined by Foulon [10], Martínez *et al* [11–13] and others is precisely the horizontal component of the classical Jacobi equation for geodesics on E. The horizontal component is shown to imply the vertical component in the case where the Jacobi field is the generator of a variation through the complete lift of a family of curves in the base $\mathbf{R} \times M$. This relationship between the classical and generalized Jacobi equations allows us to generalize a result of Martínez *et al* in theorem 5: if the generalized Jacobi equation is satisfied on a solution curve of the SODE, there is an extension of its complete lift to a symmetry of the SODE field Γ in a tubular neighbourhood of the lift of the solution curve.

At this stage, we can identify three directions deserving further investigation. The first is to extend the compact notation used in [11–13] to higher-order jet bundles, in the hope of carrying forward a classification such as in [13] for higher-order systems of differential equations. It is not at all clear how this could be done from the standpoint of [11, 12], but the approach used in this paper makes the direction clear. This has now been completed [1].

A second possibility is to use the geodesic property of lifted solutions of the SODE to write a Raychaudhuri equation (see, for example, [6]) for an arbitrary SODE. To do this in the usual way, writing the rate of change of expansion in terms of rotation and shear, would require a metric on E preserved by the linear connection $\tilde{\nabla}$. In the absence of such a metric, it is still possible to write a Raychaudhuri-like equation via a decomposition into diagonal and trace-free components. This issue will be addressed in a future publication.

The third direction lies from using the linear connection $\tilde{\nabla}$ to calculate secondary invariance classes of projective transformations of E. Since the solutions of the SODE are contained in the set of geodesics of $\tilde{\nabla}$, which are invariant under projective transformations, this could reveal the large-scale structure of the solutions of a given SODE. Related to this, the identification of the generalized Jacobi equation as a component of the conventional Jacobi equation opens the possibility of establishing an analogue of the Morse index theorem for solutions of an arbitrary SODE.

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